M250 Briggs 2.1 The Idea of Limits

Objectives

1) Evaluate and interpret a position function

2) Calculate average velocity

3) Estimate instantaneous velocity by using successive approximations of average velocity

4) Use limit notation to describe the relationship between average and instantaneous velocity.

5) Express average velocity graphically using slope of a secant line to the position function.

6) Express instantaneous velocity graphically using slope of a tangent line to the position function.

7) Use limit notation correctly.

8) Review of rounding, avoiding roundoff error, using GC effectively.

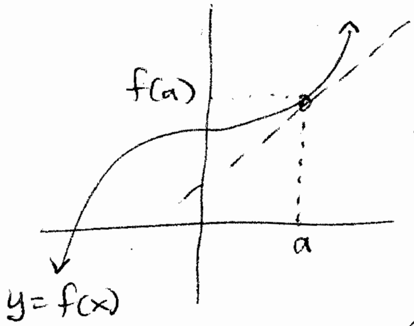
Math 250 2.1 Limits — Preview of Calculus.

What is a limit and why do we care?

Because the <u>two main calculus questions</u> are based on limits, which extend our understanding.

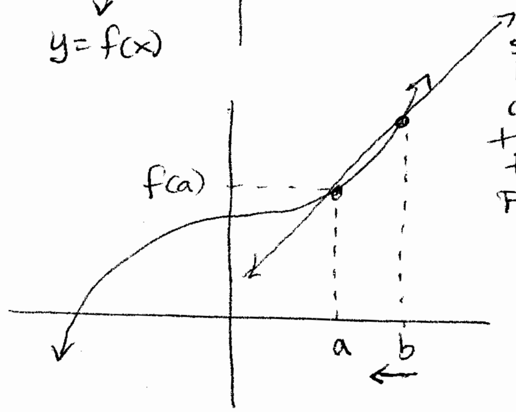
First calculus concept: slope of tangent line

(I) What is the equation of a line tangent to a curve at a single point?



y = mx + b tangent line touches only once

Need the <u>slope</u>, which we calculate from <u>two</u> points. But we only have one point.



secant line cuts through two points.

So we add a moveable second point $(b, f(b))$ and move it (take a limit) as $b \to a$.

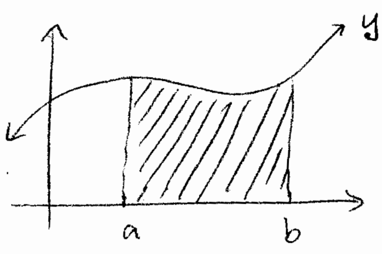
$$\frac{f(a)-f(b)}{a-b} = m_{secant}.$$

$$\lim_{b \to a} \frac{f(a)-f(b)}{a-b} = m_{tangent}$$

The slope of the secant line is an approximation of the slope of the tangent line. If b is far from a, this will be an unreliable approximation. But if b is close to a, it will be a good approximation.
When we take the limit, the approximation becomes exact.
wow!

Second calculus concept: area under curve

(II) What is the area of the plane region between a curve and the x-axis? (over a given interval).



Here we approximate the area by dividing it into small rectangles (approximation) then take the limit to get an infinite number of very tiny rectangles (exact).

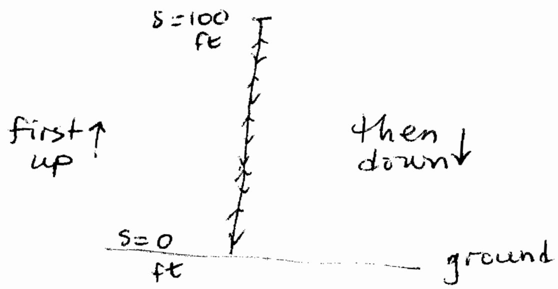
Specific example of the slope of a tangent line is <u>average velocity</u>, but we need some other ideas first.

① A rock is launched vertically upward from the ground with a speed of 80 ft/sec. Neglecting air resistance, the position of the rock after t seconds is given by

$$s(t) = -16t^2 + 80t$$

where position s is the height in feet

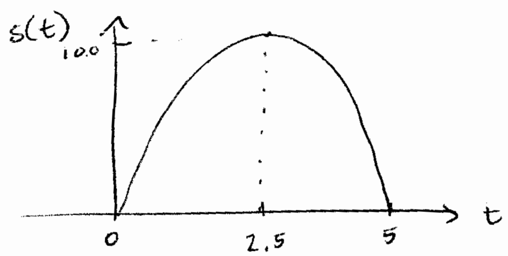
a) Draw a picture of the actual motion of the rock.



up, then down, in a straight line.

Motion in a straight line is called <u>rectilinear motion</u>.

b) Sketch the graph of s(t).



vertex of parabola $\frac{-b}{2a} = \frac{-80}{2(-16)} = 2.5 = t$

$s(2.5) = 100$

vertex $(2.5, 100)$

c) Describe the position of the rock in words.

It starts on the ground at time 0 sec, reaches its highest point 100 ft at time 2.5 sec, then falls back down, landing back on the ground at 5 sec.

d) Find the height of the rock at 1 sec, 2 sec & 2.5 sec.

$s(1) = -16(1)^2 + 80(1) = \boxed{64 \text{ ft}}$

$s(2) = -16(2)^2 + 80(2) = \boxed{96 \text{ ft}}$

$s(2.5) = \boxed{100 \text{ ft}}$

e) Find the average velocity of the rock over the time interval $[1,2]$.

Definition

average velocity over a time interval $[t_0, t_1]$	$V_{av} = \dfrac{s(t_1) - s(t_0)}{t_1 - t_0}$	change in position Δs	change in time Δt

$$V_{av} = \frac{s(2) - s(1)}{2 - 1} = \frac{96 - 64 \text{ ft}}{1 \text{ sec}} = \boxed{32 \text{ ft/sec}}$$

f) Over the entire interval from $t = 1$ sec to $t = 2$ sec, does the rock move at the same speed of 32 ft/sec?

No — the rock slows down as it goes higher because gravity pulls on it.

g) Find the average velocity of the rock over the time interval $[2, 2.5]$

$$V_{av} = \frac{s(2.5) - s(2)}{2.5 - 2} = \frac{100 - 96}{.5} = \frac{4}{2.5} = \boxed{1.6 \text{ ft/sec}}$$

Much slower than $[1,2]$

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* Rounding, Round-off Error, Using GC [Y=] and [Y-VARS]

Q: If performing a calculation involving two function evaluations, when do you round?

A: At the end and ONLY at the end.
DO NOT ROUND INTERMEDIATE CALCULATIONS EVER!

ex: correct calculation $\dfrac{1999.9999999}{0.0014999} \approx 1{,}333{,}422.228$

round to 3 places (intermediate) $\dfrac{2000.000}{0.001} = 2{,}000{,}000.$ wrong calculation.

result is off by more than 600,000.

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To do the entire calculation from part (g) in GC without intermediate rounding

→ Do entire calculation in one step. ←

- either use lots of parentheses (correctly)
- or use $\boxed{Y=}$ and $\boxed{\text{VARS}}$

step 1: Enter the function s(t) into $\boxed{Y=}$ menu using x instead of t.

$\boxed{Y=}$

$Y_1 = -16x^2 + 80x$

step 2: From basic calculation screen type

$(Y_1(2.5) - Y_1(2))\ /\ (2.5 - 2)$

parentheses around numerator & denominator are required!

$Y_1(2.5)$ is function notation

GC knows ( ) means "evaluate."

To access Y_1 Press $\boxed{\text{VARS}}$

▷ to $\boxed{\text{Y-VARS}}$

select 1: Function

select 1: Y_1

TI-84 CE can use $\boxed{\text{ALPHA}}$ $\boxed{\text{F4}}$ instead

h) Estimate the instantaneous velocity of the rock at t=2 by calculating average velocities over the following intervals

B A		B A	
[2, 2.5]	8ft/sec	[1.5, 2]	24 ft/sec
[2, 2.25]	12ft/sec	[1.75, 2]	20 ft/sec
[2, 2.1]	14.4 ft/sec	[1.9, 2]	17.6 ft/sec
[2, 2.01]	15.84 ft/sec	[1.99, 2]	16.16 ft/sec
[2, 2.001]	15.984 ft/sec	[1.999, 2]	16.016 ft/sec
[2, 2.0001]	15.9984 ft/sec	[1.9999, 2]	16.0016 ft/sec
[2, 2.00001]	15.99984 ft/sec	[1.99999, 2]	16.00016 ft/sec

42 stu ⇒ 14×3 ⇒ 3 Students per interval

GC next page!

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To do many calculations with similar structure on GC:

Option 1: Use $\boxed{\text{2nd}}$ $\boxed{\text{ENTER}}$ = $\boxed{\text{ENTRY}}$ to recall most recent calculation

(Type $\boxed{\text{2nd}}$$\boxed{\text{ENTER}}$ again to get the calculation before that.

Can repeat up to 10–15 times, depending on operating system.)

Then $\boxed{\text{DEL}}$ to delete old info

and $\boxed{\text{2nd}}$ $\boxed{\text{DEL}}$ = $\boxed{\text{INS}}$ (insert) to replace with new info.

When calculation looks like desired calculation, press $\boxed{\text{ENTER}}$.

Option 2: Create a Y_2 in $\boxed{Y=}$ menu which references $\boxed{Y_1}$.
then use $\boxed{\text{ASK}}$ $\boxed{\text{TABLE}}$.

Desired calculations all look like $\dfrac{s(x)-s(2)}{x-2}$ in left column

or $\dfrac{s(2)-s(x)}{2-x}$ in right column

⇒ NOTE: These are equivalent! Mult by $\frac{-1}{-1}$ and dist.

in $\boxed{Y=}$ leave $Y_1 = -16x^2 + 80x$

in $Y_2 = (Y_1(x) - Y_1(2))\,/\,(x-2)$

access as before $\boxed{\text{VARS}}$ ▶ $\boxed{\text{Y-VARS}}$
1: Function
1: Y_1

in $\boxed{\text{TBLSET}}$ Indpnt: $\boxed{\text{Ask}}$
Depend: $\boxed{\text{Auto}}$

in $\boxed{\text{2nd}}$$\boxed{\text{GRAPH}}$ = $\boxed{\text{TABLE}}$ ▶▶ to hover on Y_2 and see all decimals displayed at bottom of screen. Then ▼

x	Y_1	Y_2
2.5		8
2.25		12
2.1	ignore	14.4
2.01		15.84
2.001		15.984
2.0001		15.9984
2.00001		15.99984

repeat

x	Y_1	Y_2
1.5		24
1.75		20
1.9	ignore	17.6
1.99		16.16
1.999		16.016
1.9999		16.0016
1.99999		16.00016

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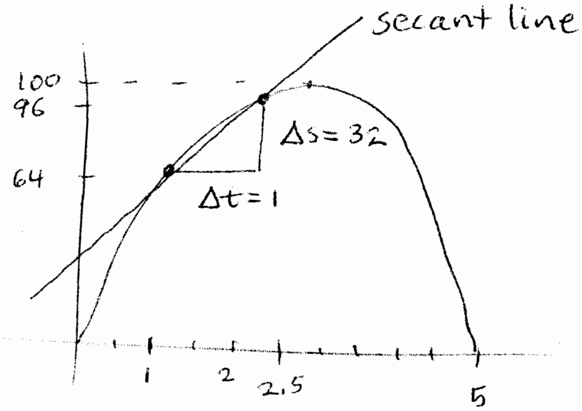
The average velocities as our time intervals get smaller (yet near $t=2$) seem to be approaching 16 ft/sec.

$\boxed{V_{inst} \approx 16 \text{ ft/sec}} \leftarrow \lim\limits_{t \to 2} \dfrac{s(t)-s(2)}{t-2}$

i) Draw a line related to the average velocity on [1,2] together with the graph of $s(t)$

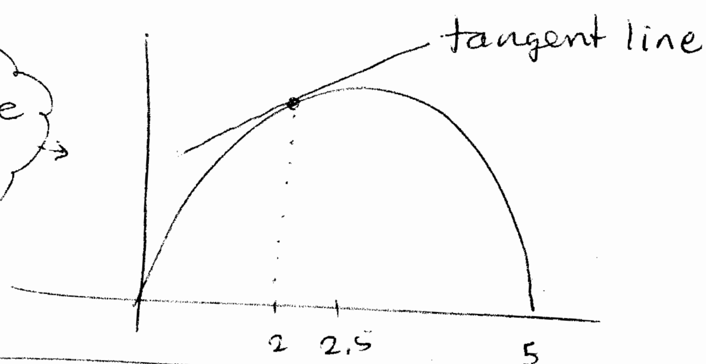


A line that crosses through 2 pts of another graph is called a secant line (from Latin "secare," to cut)

* The average velocity on [1, 2] is the slope of the secant line through $(1, s(1))$ and $(2, s(2))$.

j) Draw a line which is related to the instantaneous velocity at $t = 2$.

e-book has awesome interactive diagram



A line which touches the graph at one point (in such a way that zooming in on the line looks a lot like the graph) is called a tangent line (from Latin "tangere", to touch)

* The instantaneous velocity on [1,2] is the slope of the tangent line at $(2, s(2))$.

We have only the one point $(2, s(2))$, so we cannot calculate the slope of the tangent line as a single calculation.

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We estimated the slope of the tangent line using successive approximations of the slope of the secant line.

We will make this process more formal in later lessons, but for now, let's give it a name and some notation

$$\boxed{\begin{array}{l} \underset{\text{on } [t_0, t_1]}{V_{av}} = m_{sec} = \dfrac{s(t_1) - s(t_0)}{t_1 - t_0} \\ \\ \underset{\substack{\text{at} \\ t = t_0}}{V_{inst}} = m_{tan} = \displaystyle\lim_{t_1 \to t_0} \dfrac{s(t_1) - s(t_0)}{t_1 - t_0} \end{array}}$$

The limit notation $\displaystyle\lim_{t_1 \to t_0}$ is never used by itself, because it means "take the limit of" the expression which follows.

NO

The limit notation should also be removed when the limit has been found:

Yes $\displaystyle\lim_{t \to 2} \underbrace{\frac{s(t) - s(2)}{t - 2}}_{\substack{\text{The argument} \\ \text{of the limit} \\ \text{notation.}}} = \underbrace{16 \text{ ft/sec}}_{\text{no limit notation}}$

Do <u>not</u> write $\displaystyle\lim_{t \to 2} = 16$. This is nonsense.

No

Similarly, don't leave off the limit notation

No $\underbrace{\dfrac{s(t) - s(2)}{t - 2}}_{\text{This is an expression with t.}} = 16 \text{ ft/sec}$ is <u>also</u> nonsense!

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But back to part h).

Notice that the t-values that were > 2 lead to the $v_{inst} = 16$

AND the t-values < 2 also lead to $v_{inst} = 16$.

What if the values to the right $(t > 2)$ led to a different v_{inst} from the one we get using values to the left $(t < 2)$?

If this happens we say

$$\lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} \quad \text{does not exist (or DNE)}$$

OR, if we want to be specific:

$$\lim_{t \to 2^+} \frac{s(t) - s(2)}{t - 2}$$ is the one-sided limit from the right "right-sided limit"

$$\lim_{t \to 2^-} \frac{s(t) - s(2)}{t - 2}$$ is the one-sided limit from the left "left sided limit"

If the limit exists, it must be the same from left or right.

$$\lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2^+} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2^-} \frac{s(t) - s(2)}{t - 2}$$